

Formal Homotopy Quantum Field Theories

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Joint work with V. Turaev.

Plan.

- Recall of Crossed modules, etc.
- Topological Quantum Field Theories
- Homotopical Quantum Field Theories
- Classification results: (i) $B = K(\pi, 1)$, (ii) $B = K(A, 2)$.
- Formal Maps and Formal HQFTs : general results
- Crossed \mathcal{C} -algebras.

1. Crossed Modules, etc

Definition

A *crossed module*, $\mathcal{C} = (C, P, \partial)$, consists of groups C, P , a (left) action of P on C (written $(p, c) \rightarrow {}^p c$) and a homomorphism

$$\partial : C \rightarrow P$$

such that

$$\text{CM1} \quad \partial({}^p c) = p \cdot \partial c \cdot p^{-1} \quad \text{for all } p \in P, c \in C;$$

and

$$\text{CM2} \quad \partial c c' = c \cdot c' \cdot c^{-1} \quad \text{for all } c, c' \in C.$$

Examples

- If N is a normal subgroup of a group P , then P acts by conjugation on N , ${}^p n = pnp^{-1}$, and the inclusion $\iota : N \rightarrow P$ is a crossed module.
- If M is a left P -module and we define $0 : M \rightarrow P$ to be the trivial homomorphism, $0(m) = 1_P$, for all $m \in M$, then $(M, P, 0)$ is a crossed module.
- If G is any group, $\alpha : G \rightarrow \text{Aut}(G)$, the canonical map sending $g \in G$ to the inner automorphism determined by g , is a crossed module for the obvious action of $\text{Aut}(G)$ on G .

Also for an algebra, L , $(U(L), \text{Aut}(L), \delta)$ is a crossed module, where $U(L)$ is the group of units of L and δ maps a unit e to the automorphism given by conjugation by e .

Examples continued

- Central extension: $1 \rightarrow A \rightarrow E \xrightarrow{\partial} P \rightarrow 1$,
 $E \xrightarrow{\partial} P$ is a crossed module.

- Fibration:

$$F \rightarrow E \rightarrow B \quad \text{pointed spaces}$$

$\pi_1(F) \rightarrow \pi_1(E)$ is a crossed module.

- Special case: (X, A) pointed pair of spaces, $\pi_2(X, A) \rightarrow \pi_1(A)$ is a crossed module.
- Extra special case: X , CW-complex, $A = X_1$, 1-skeleton of X ,

$$\pi_2(X, X_1) \xrightarrow{\partial} \pi_1(X_1)$$

determines 2-type of X . NB. $\ker \partial = \pi_2(X)$, $\text{coker} \partial = \pi_1(X)$.

2. TQFTs (over \mathbb{C}).

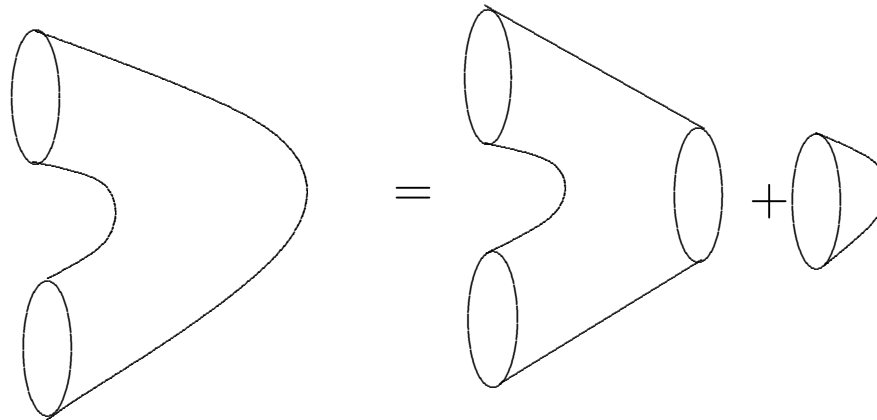
$\{\text{Oriented } d\text{-manifolds, } X\} \rightarrow \{\text{f.d. Vector spaces, } T(X)\}$

$\{\text{cobordisms, } M : X \rightarrow Y\} \rightarrow \{\text{lin. trans., } T(M) : T(X) \rightarrow T(Y)\}$

'tensor' of manifolds is disjoint union, $X \sqcup Y$

\rightarrow usual tensor, $T(X) \otimes T(Y)$

Compose cobordisms : usual picture with usual provisos on associativity etc.



Get that a TQFT is simply a monoidal functor:

$$\begin{array}{c} (d - \text{cobord}, \sqcup) \\ \downarrow T \\ (\text{Vect}, \otimes) \end{array}$$

NB. \emptyset is a d -manifold so a closed $d + 1$ manifold M is a cobordism from \emptyset to \emptyset and hence $T(M) : T(\emptyset) \rightarrow T(\emptyset)$. \emptyset is the unit for the monoidal structure on d -cobord, and the ground field \mathbb{C} is that for Vect. As T is monoidal, $T(\emptyset) \cong \mathbb{C}$, so $T(M)$ is a linear map from \mathbb{C} to itself so is specified by a single complex number, a numerical invariant of M

3. HQFTs

Problem : would like to have a theory with manifolds **with extra structure**, e.g. a given G -bundle, metric etc.

Suggestion by Turaev (1999): Replace X by characteristic structure map, $g : X \rightarrow B$, where B is a 'background' space, for instance, $B = BG$, the classifying space of a group G .

For the cobordisms, want $F : M \rightarrow B$ agreeing with the structure maps on the ends but F will only be given 'up to homotopy relative to the boundary'.

Get a monoidal category $d\text{-Hocobord}(B) : (\text{Rodrigues, 2000})$

Definition: **A HQFT is a monoidal functor**

$$d\text{-Hocobord}(B) \rightarrow \text{Vect}.$$

4. Classification results for cases where: (i) $B = K(\pi, 1)$, (ii) $B = K(A, 2)$.

General result (Rodrigues) :

d -dimensional HQFTs over B depend on the $(d + 1)$ -type of B only, so can assume $\pi_n(B)$ is trivial for $n > d + 1$.

Clearest classification results are in dimensions $d = 1$ and $d = 2$.
(We will look at $d = 1$ only.)

- d -manifold = disjoint union of oriented circles,
- cobordism = oriented surface possibly with boundary,
- can restrict to B being a 2-type.

Previously known cases:

(i) $B = K(\pi, 1)$, so $\pi_1(B) = \pi$, $\pi_i(B) = 1$ for $i > 1$, and ‘extra structure’ = ‘isomorphism class of principal π -bundles’.

(Turaev, 1999)

HQFTs over $K(\pi, 1) \longleftrightarrow$

π -graded algebras with inner product and ‘extra structure’

= crossed π -algebras.

Crossed π -algebras.

$$L = \bigoplus_{g \in \pi} L_g, \quad \text{a } \pi\text{-graded algebra}$$

- if ℓ_1 is graded g , and ℓ_2 is graded h , then $\ell_1 \ell_2$ is graded gh ;
- L has a unit $1 = 1_L \in L_1$ for 1 , the identity element of π ;

- there is a symmetric K -bilinear form

$$\rho : L \otimes L \rightarrow K$$

such that

- (i) $\rho(L_g \otimes L_h) = 0$ if $h \neq g^{-1}$;
- (ii) the restriction of ρ to $L_g \otimes L_{g^{-1}}$ is non-degenerate for each $g \in \pi$, (so $L_{g^{-1}} \cong L_g^*$, the dual of L_g);
and
- (iii) $\rho(ab, c) = \rho(a, bc)$ for any $a, b, c \in L$.

- a group homomorphism

$$\phi : \pi \rightarrow \text{Aut}(L)$$

satisfying:

- (i) if $g \in \pi$ and we write $\phi_g = \phi(g)$ for the corresponding automorphism of L , then ϕ_g preserves ρ , (i.e. $\rho(\phi_g a, \phi_g b) = \rho(a, b)$) and

$$\phi_g(L_h) \subseteq L_{ghg^{-1}}$$

for all $h \in \pi$;

- (ii) $\phi_g|_{L_g} = id$ for all $g \in \pi$;

- (iii) for any $g, h \in \pi$, $a \in L_g$, $b \in L_h$, $\phi_h(a)b = ba$;

- (iv) for any $g, h \in \pi$ and $c \in L_{ghg^{-1}h^{-1}}$,

$$\text{Tr}(c\phi_h : L_g \rightarrow L_g) = \text{Tr}(\phi_{g^{-1}}c : L_h \rightarrow L_h),$$

where Tr denotes the K -valued trace of the endomorphism.

(Brightwell and Turner, 2000)

HQFTs over $K(A, 2) \longleftrightarrow$

Frobenius algebras with an action of A
= A -Frobenius algebras.

VT and TP, 2003-2005.

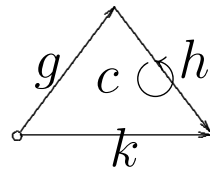
General 2-type B given by a crossed module $\mathcal{C} = (C, P, \partial)$. Can one find corresponding algebras and ways of studying them hopefully leading to a classification theorem for HQFTs over $B\mathcal{C}$.

Progress so far: Notion of Formal HQFT based on combinatorial models of characteristic maps. Crossed \mathcal{C} -algebras giving classification for this formal model.

4. Formal \mathcal{C} - maps and formal HQFTs

Formal \mathcal{C} - map on a manifold M :

- decomposition of manifold M into cells, e.g. a triangulation.
- labelling of edges with elements of P ;
- labelling of faces with elements of C ;
- boundary condition:



with $g, h, k \in P$ and $c \in C$, and where $\partial c = kh^{-1}g^{-1}$.

- cocycle condition from any 3-simplices.

This correspond combinatorially to the characteristic map

$$g : M \rightarrow BC.$$

A formal HQFT with background \mathcal{C} assigns

- to each formal \mathcal{C} -circuit, $\mathbf{g} = (g_1, \dots, g_n)$, a K -vector space $\tau(\mathbf{g})$, and by extension, to each formal \mathcal{C} -map on a 1-manifold S , given by a list, $\mathbf{g} = \{\mathbf{g}_i \mid i = 1, 2, \dots, m\}$ of formal \mathcal{C} -circuits, a vector space $\tau(\mathbf{g})$ and an isomorphism,

$$\tau(\mathbf{g}) = \bigotimes_{i=1, \dots, m} \tau(\mathbf{g}_i),$$

identifying $\tau(\mathbf{g})$ as a tensor product;

- to any formal \mathcal{C} -cobordism, (M, \mathbf{F}) between (S_0, \mathbf{g}_0) and (S_1, \mathbf{g}_1) , a K -linear transformation

$$\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \rightarrow \tau(\mathbf{g}_1),$$

These assignments are to satisfy the following axioms:

(i) Disjoint union of formal \mathcal{C} -maps corresponds to tensor product:

$$\tau(\mathbf{g} \sqcup \mathbf{h}) \xrightarrow{\cong} \tau(\mathbf{g}) \otimes \tau(\mathbf{h}), \quad \tau(\emptyset) \xrightarrow{\cong} \mathbb{C}.$$

(ii) For formal \mathcal{C} -cobordisms

$$\mathbf{F} : \mathbf{g}_0 \rightarrow \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \rightarrow \mathbf{g}_2$$

with composite $\mathbf{F} \#_{\mathbf{g}_1} \mathbf{G}$, we have

$$\tau(\mathbf{F} \#_{\mathbf{g}_1} \mathbf{G}) = \tau(\mathbf{G})\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \rightarrow \tau(\mathbf{g}_2).$$

(iii) For the identity formal \mathcal{C} -cobordism on \mathbf{g} ,

$$\tau(1_{\mathbf{g}}) = 1_{\tau(\mathbf{g})}.$$

(iv) Interaction of cobordisms and disjoint union is transformed correctly by τ .

Definition

Let $\mathcal{C} = (C, P, \partial)$ be a crossed module. A *crossed \mathcal{C} -algebra* consists of a crossed P -algebra, $L = \bigoplus_{g \in P} L_g$, together with elements $\tilde{c} \in L_{\partial c}$, for $c \in C$, such that

(a) $\tilde{1} = 1 \in L_1$;

(b) for $c, c' \in C$, $\widetilde{(c'c)} = \tilde{c}' \cdot \tilde{c}$;

(c) for any $h \in P$, $\phi_h(\tilde{c}) = \tilde{h}c$.

We note that the first two conditions make ‘tilderisation’ into a group homomorphism $(\sim) : C \rightarrow U(L)$, the group of units of the algebra, L .

In fact:

Suppose that L is a crossed \mathcal{C} -algebra. The diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(\sim)} & U(L) \\ \partial \downarrow & & \downarrow \delta \\ P & \xrightarrow{\phi} & \mathcal{A}ut(L) \end{array} \quad (1)$$

is a morphism of crossed modules from \mathcal{C} to $\mathcal{A}ut(L)$.

Finally:

Main theorem

Formal HQFTs over \mathcal{C} correspond to crossed \mathcal{C} -algebras up to isomorphism.

Future work?

1. Investigate change of algebras along morphisms of crossed modules, especially along weak equivalences;
2. Formal \mathcal{C} -maps specify \mathcal{C} -bundles as in Baez's lectures. (Investigate, some work has been done by R.Lewis, student UWB.)
3. Open-closed HQFTs (Moore-Segal) correspond to K-theoretic invariants of the crossed algebras for $B = K(G, 1)$. Generalise.
4. Extend the theory (i) to 2+1 dimensions and then to formal coefficients in a model for a 3-type.
5. a lot more!

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Bangor, 30 June 2005